Lecture Nanophotonics
Coupled light-matter systems & Nonlinear optics Assignment

Class dates:	*May 11th and 13th, 2020*
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If you have any questions regarding to the exercise, you can contact the teaching assistants through email. Please hand in your finished assignments through email.

1 Coupled harmonic oscillators

Consider the equations of motion of two coupled damped oscillators 1 and 2:

$$\begin{aligned} \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \gamma_1 \frac{\mathrm{d}x_1}{\mathrm{d}t} + \omega_1^2 x_1 - \Omega^2 x_2 = 0 \\ \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \gamma_2 \frac{\mathrm{d}x_2}{\mathrm{d}t} + \omega_2^2 x_2 - \Omega^2 x_1 = 0 \end{aligned} \tag{1}$$

Under certain approximations and assuming solutions of the form: $x_{1,2} = C_{1,2}\, e^{-i\omega t}$, the above system can be mapped into an eigenvalue problem $|H - \omega I| = 0$ with a non-Hermitian Hamiltonian:

$$H = \begin{pmatrix} \omega_1 - i\frac{\gamma_1}{2} & -\frac{\kappa}{2} \\ -\frac{\kappa}{2} & \omega_2 - i\frac{\gamma_2}{2} \end{pmatrix} \tag{2}$$

(a) Show that the eigenvalues of this Hamiltonian, which correspond to the eigenfrequencies of the coupled oscillators, can be expressed as $\omega_\pm = \bar{\omega} \pm \frac{1}{2}\sqrt{\Delta^2 + \kappa^2}$, where $\bar{\omega} = \frac{\omega_1+\omega_2}{2} - i\frac{\gamma_1+\gamma_2}{4}$, $\Delta = \omega_1 - \omega_2 - i\frac{\gamma_1-\gamma_2}{2}$, $\kappa = \frac{\Omega^2}{\bar{\omega}}$ (assume $\omega_1 \approx \omega_2 \approx \omega$).

(b) Assuming $\omega_1 = 1$, $\gamma_1 = 0.01$, $\gamma_2 = 0.02$, plot $\mathrm{Re}\{\omega_\pm\}$ and $\mathrm{Im}\{\omega_\pm\}$ as a function of ω_2 in the range $\omega_2 \in [0.995, 1.005]$ at three κ: $\kappa = 0.004, 0.005, 0.006$ (plot $\mathrm{Re}\{\omega_\pm\}$ in one figure, $\mathrm{Im}\{\omega_\pm\}$ in another figure).

(c) Assuming $\omega_1 = \omega_2 = 1$, $\gamma_1 = 0.01$, $\gamma_2 = 0.02$, plot $\mathrm{Re}\{\omega_\pm\}$ and $\mathrm{Im}\{\omega_\pm\}$ as a function of κ in the range $\kappa \in [0, 0.01]$ (plot $\mathrm{Re}\{\omega_\pm\}$ in one figure, $\mathrm{Im}\{\omega_\pm\}$ in another figure). Find the κ_c at which $\mathrm{Re}\{\omega_+\} = \mathrm{Re}\{\omega_-\}$ and $\mathrm{Im}\{\omega_+\} = \mathrm{Im}\{\omega_-\}$.

In non-Hermitian quantum mechanics, the condition where $\mathrm{Re}\{\omega_+\} = \mathrm{Re}\{\omega_-\}$ and $\mathrm{Im}\{\omega_+\} = \mathrm{Im}\{\omega_-\}$ is known as the **exceptional point**.
Now consider a driving force $F\, e^{-i\omega t}$ acting on oscillator 1. The equations of motion become:

$$\begin{aligned} \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \gamma_1 \frac{\mathrm{d}x_1}{\mathrm{d}t} + \omega_1^2 x_1 - \Omega^2 x_2 = F e^{-i\omega t} \\ \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \gamma_2 \frac{\mathrm{d}x_2}{\mathrm{d}t} + \omega_2^2 x_2 - \Omega^2 x_1 = 0 \end{aligned} \tag{3}$$

(d) Assuming $x_{1,2} = C_{1,2} e^{-i\omega t}$, prove the expressions for the complex amplitudes $C_{1,2}$ are:

$$\begin{aligned} C_1 = \frac{(\omega_2^2 - \omega^2 - i\omega\gamma_2)F}{(\omega_2^2 - \omega^2 - i\omega\gamma_2)(\omega_1^2 - \omega^2 - i\omega\gamma_1) - \Omega^4} \\ C_2 = \frac{\Omega^2 F}{(\omega_2^2 - \omega^2 - i\omega\gamma_2)(\omega_1^2 - \omega^2 - i\omega\gamma_1) - \Omega^4} \end{aligned} \tag{4}$$

(e) The complex amplitude $C_{1,2}$ can be expressed in terms of its modulus and phase as $C_{1,2} = |C_{1,2}|\, e^{i\phi_{1,2}}$. Assuming $\omega_1 = 1$, $\omega_2 = 1.05$, $\gamma_1 = 0.01$, $\gamma_2 = 0.001$ and $\Omega = 0.2$, make the following plots in the range $\omega \in [0.9, 1.1]$:

(i) $|C_1|$ and $|C_2|$;

(ii) ϕ_1, ϕ_2 and $\phi_1 - \phi_1$;

Explain what happens to $|C_1|$ at ω_2, based on the behavior of the phases of the coupled oscillators. (Hint: think of what interferes with what.)

2 Single non-linear oscillator

The field α of a single-mode nonlinear optical cavity, driven by a monochromatic field of frequency ω and amplitude F, is described by the following equation (assuming $\hbar = 1$):

$$i\dot{\alpha} = \left(\omega_0 - i\frac{\gamma}{2}\right)\alpha + U|\alpha|^2\alpha + Fe^{-i\omega t}. \tag{5}$$

Here, ω_0 is the cavity resonance frequency, $\gamma/2$ is the energy loss rate, and U is the photon-photon interaction strength leading to optical nonlinearity. For convenience, let us move to a frame rotating at the driving frequency ω, such that Eq.(5) becomes

$$i\dot{\alpha} = \left(-\Delta - i\frac{\gamma}{2}\right)\alpha + U|\alpha|^2\alpha + F. \tag{6}$$

where $\Delta = \omega - \omega_0$ is the detuning between the driving field and the cavity resonance. We are interested in the steady-state solution(s) to Eq. 6, given by setting $\dot{\alpha} = 0$.

(a) Show that the steady-state solution satisfies the following equation:

$$|F|^2 = U^2N^3 - 2\Delta UN^2 + \left(\Delta^2 + \frac{\gamma^2}{4}\right)N, \tag{7}$$

where $N = |\alpha|^2$ is the steady-state density (number of photons in the cavity).

(b) Assuming $\gamma = 0.01$, $U = 0.0075\gamma$, and $\omega_0 = 1$, use Eq. 7 to make a plot of N vs. $|F|^2$ in the range $(|F|/\gamma)^2 \in [23, 30]$ at two different Δ: i) $\Delta = 0.85\gamma$ and ii) $\Delta = 0.90\gamma$. Plot the two curves in the same figure using the computational tool of your choice.

(c) As shown in the plots of (b), for a sufficiently large Δ there are three steady states for certain F. Derive an analytical expression for the critical detuning Δ_c needed to observe three steady-states at certain F. (Hint: think of $\partial|F|^2/\partial N$)

(d) Here you will show through a linear stability analysis that, in the regime where three steady-states exist, only two of the three steady-states are stable. Hence, this regime is known as "bistability". To make the stability analysis, follow this procedure:

(i) Add a small fluctuation $\delta\alpha$ to the steady-state solution, i.e. let $\tilde{\alpha} = \alpha + \delta\alpha$, and plug this perturbed solution in Eq. 6. Then, expand that resulting equation and retain only the terms which are linear in the fluctuation $\delta\alpha$. Show that the differential equation for the fluctuation $\delta\alpha$ and its complex conjugate $\delta\alpha^*$ can be written in matrix form:

$$i\frac{\partial}{\partial t}\begin{pmatrix}\delta\alpha \\ \delta\alpha^*\end{pmatrix} = \mathbf{A}\begin{pmatrix}\delta\alpha \\ \delta\alpha^*\end{pmatrix} \tag{8}$$

with

$$\mathbf{A} = \begin{pmatrix}-\Delta - \frac{i\gamma}{2} + 2UN & U\alpha^2 \\ U\alpha^{*2} & -\Delta - \frac{i\gamma}{2} - 2UN\end{pmatrix} \tag{9}$$

(ii) Differential equations of this form have solutions $\boldsymbol{\delta\alpha}(t) = \boldsymbol{\eta}\exp^{-i\lambda t}$, with λ the eigenvalues of the matrix $\boldsymbol{A}$. This implies that if there exists an eigenvalue for which $\text{Im}(\lambda_j) \geq 0$, then fluctuations will grow in time and hence the steady-state solution is unstable. Based on this, make the stability analysis of the steady-state solutions you found in part (b)(ii), and remake the plots now properly labeling the unstable solutions with a different color or line style.