Lecture Nanophotonics
Plasmonics and Metamaterials Assignment

Handout date:	Wednesday, April 1 2020
Due date:	Monday, April 13 2020
Teaching assistants:	Isabelle Palstra (i.palstra@amolf.nl)
	Tom Wolterink t.wolterink@amolf.nl)

Note:

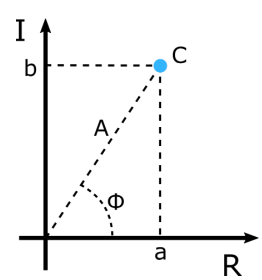
- We will only work with SI units.
- We use the convention in which the time-dependent part of a time-harmonic variable $f(\omega)$ is given as $e^{-i\omega t}$, with ω the angular frequency. For instance, $E(t) = Ee^{-i\omega t}$.

1 Fresnel reflection coefficient (p-polarization)

Consider an interface between two media 1 and 2 with dielectric constants $\epsilon_1 = 2.25$ and $\epsilon_2 = 1$, respectively. The magnetic permeabilities are equal to one. A p-polarized plane wave with wavelength $\lambda = 532$ nm is incident from medium 1 at an angle of incidence of θ_1. P-polarized means that the magnetic field is parallel to the interface. See slide 49 from Monday's lecture for the s-polarization.

a) Derive the Fresnel reflection coefficient r if $E_r = rE_{\text{in}}$, where E_{in}, E_r are the incoming and reflected electric fields, and r is the reflection coefficient. Start by writing down the conservation laws for the parallel components of the **E** and **H** fields.

b) Remember that any complex number C can be expressed in two ways; the sum of the real and imaginary part, and as an amplitude and phase, $C = a + ib = A\mathrm{e}^{i\Phi}$.
Express the reflection r in terms of amplitude A and phase Φ ($r = Ae^{i\Phi}$) and plot A and Φ as a function of θ_1. What are the consequences for the reflected wave?



2 Evanescent field

An evanescent field is an oscillating electric and/or magnetic field whose energy is spatially concentrated around a point or an interface. It can be described by plane waves ($\mathbf{E}e^{\mathrm{i}(\mathbf{k}r-\omega t)}$) with at least one component of the wavevector ($\mathbf{k}$) is imaginary. Total internal reflection is a typical way to generate an evanescent field.

a) A plane wave is incident on a water(n=1.33)-air interface at a 60° angle with the surface normal (z-axis). Show that in air, the z-component of the wavevector is imaginary.

b) An imaginary wave vector implies that a wave does not propagate, but exponentially decays. What is the decay length in the z-direction? These evanescent fields are frequently used in microscopy, as shown in Fig 1. Compare the spatial extent at a wavelength of 632 nm (HeNe laser) to the size of a bacterium. What does this tell you?

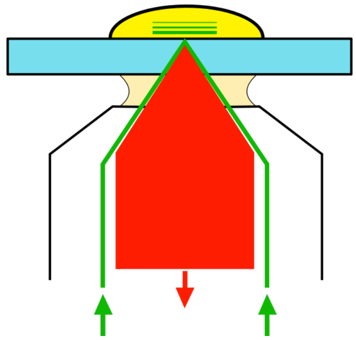


Figure 1: Total internal reflection microscopy. The sample (yellow) is illuminated by an excitation beam under an angle (green), so that the beam reflects off the glass-air interface. The evanescent field penetrates the sample and makes the sample fluoresce. The fluorescence (red) is again captured by the objective.

3 Kramers-Kronig relations

The Kramers-Kronig equations constitute an important relation between the real and imaginary part of any linear response function in a physical system. They can be derived using only causality, and are therefore of very general validity.
Assume a system in which a quantity P (e.g. macroscopic polarization) is related to an external quantity E (e.g. electric field) by a linear response function χ in the frequency domain:

$$P(\omega) = \chi(\omega)\, E(\omega) \tag{1}$$

Then, in the time domain:

$$P(t) = \int_{-\infty}^{-\infty} \chi(t-t')\, E(t')dt' \tag{2}$$

Because this is a physical system, in the time domain it must hold that:

$$\{P(t), \chi(t), E(t)\} \in \mathbb{R} \qquad \text{for all } t. \tag{3}$$

a) Discuss briefly why causality implies that

$$\chi(t) = 0 \qquad \text{for} \qquad t \leq 0. \tag{4}$$

b) Write $\chi(\omega)$ in terms of $\chi(t)$ and split in its real and imaginary part $\chi'(\omega)$ and $\chi''(\omega)$.

Now we will use a mathematical trick to simplify the previous result. As with any function, $\chi(t)$ can be expressed in its even and odd parts $\chi_e(t)$ and $\chi_o(t)$ as

$$\chi(t) = \chi_e(t) + \chi_o(t). \tag{5}$$

Using the condition in Eqn. 4, it holds that

$$\chi_e(t) = \text{sgn}(t)\chi_o(t). \tag{6}$$

This is easy to check. In Figure 2 we show an example of a function that obeys causality, that has been split into it's even and odd parts. Here, we have shown the function

$$\chi(t) = \Theta(t) \cdot \mathrm{e}^{-\gamma t} \cdot \sin(ft), \tag{7}$$

where Θ is the unit step function. If you find this tricky to understand, imagine $\chi(t)$ is a unit step function and write down χ_e and χ_o for this case.

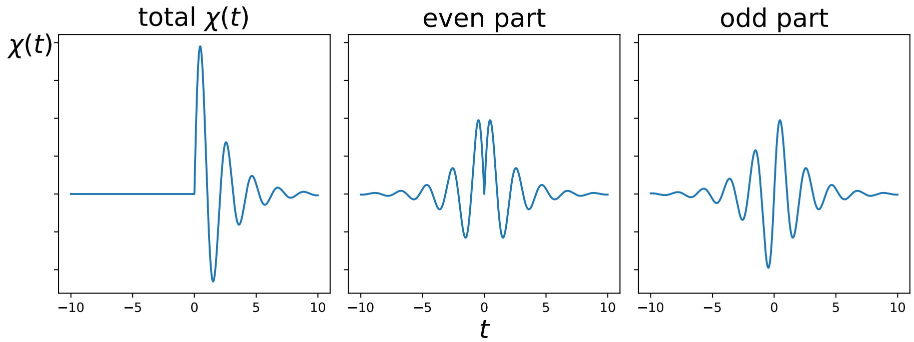


Figure 2: An example of a possible $\chi(t)$

c) Use the results from questions 3b), Eqn. 6, and the convolution theorem to show that

$$\chi'(\omega) = [i\,\chi''(\omega)] \star \left[\int_{-\infty}^{\infty} \operatorname{sgn}(t) e^{i\omega t} dt\right] \tag{8}$$

where $\star$ denotes a convolution.

Hint: First express $\chi'(\omega)$ and $\chi''(\omega)$ in terms of $\chi_o(t)$

d) Show that

$$\chi'(\omega) = \int_{-\infty}^{\infty} \frac{2\chi''(\omega')}{\omega' - \omega} d\omega' \tag{9}$$

Hint: use the relation:

$$\lim_{\epsilon \downarrow 0} \int_0^{\infty} e^{(i\omega-\epsilon)t} = \lim_{\epsilon \downarrow 0} \frac{1}{i\omega - \epsilon} e^{(i\omega-\epsilon)t}\Big|_{t=0}^{\infty} = \frac{i}{\omega} \tag{10}$$

You have derived the first of the two Kramers-Kronig relations, which relate the real and imaginary parts of physical quantities. In optics, they describe the relation between the dispersion ($\chi'(\omega)$) and the absorption ($\chi''(\omega)$) of light.

e) Is it possible for a material to have a narrow frequency window of high absorption and no dispersion? No dispersion means that $\chi'(\omega)$ is constant for all ω. If you want, you can illustrate your answer using a delta-function absorption $\chi''(\omega) = C_1\delta(\omega - \omega_0)$ with $C_1 \in \mathbb{R}$.

f) The second Kramers-Kronig relation can be derived similarly. You do not need to derive it. It is given by

$$\chi''(\omega) = -\int_{-\infty}^{\infty} \frac{2\chi'(\omega')}{\omega' - \omega} d\omega'. \tag{11}$$

Is it possible for a material to have a window of non-zero dispersion with no absorption? You may give your answer using a delta-peak shaped dispersion $\chi(\omega)'$