Nanophotonics 2019 - Problem Set Microcavities

Class date:	Monday, 20 April 2020
Due date:	Wednesday, 6 May 2020
Teaching assistants:	Cesare Paulin (paulin@amolf.nl)

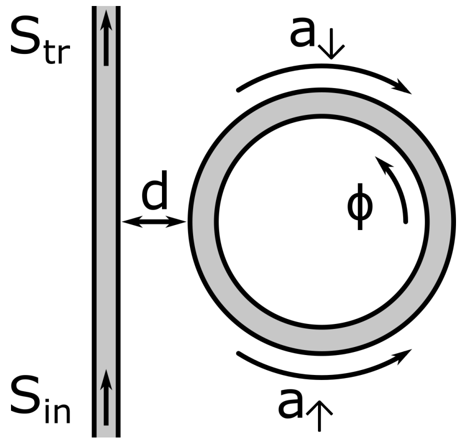

Figure 1: Schematic diagram of the situation discussed in the assignment.

An optical fibre is positioned at a distance d from a microtoroid (ring) in vacuum. Light at the resonance frequency of a whispering gallery mode (WGM) in the toroid is sent into the waveguide, as in figure 1. At $d = d_{\mathrm{crit}}$ critical coupling is reached between the fibre and the ring resonator. The toroid exhibits two whispering gallery modes, the clockwise $a_\downarrow$ and anti-clockwise $a_\uparrow$, to which we will refer with the arrow notation according to the right hand rule as in figure 1.

Problem 1 The transmitted light amplitude s_{tr} through the fibre is related to the input light amplitude s_{in} and the intracavity light amplitude $a_\downarrow$ as $s_{\mathrm{tr}} = s_{\mathrm{in}} - \sqrt{\kappa_{\mathrm{ex}}} a_\downarrow$. The light propagating forward through the fibre only couples to the clockwise-propagating WGM. This WGM corresponds to a travelling wave that, due to the imaginary exponential mode profile, only couples to forward propagating light in the fibre, so there is no reflection in the fibre.

(a) Give an expression for the transmission $T = |s_{\mathrm{tr}}|^2/|s_{\mathrm{in}}|^2$ as a function of frequency detuning Δ, coupling rate κ_{ex}, and damping rate κ.

(b) How much light is transmitted on resonance at critical coupling? What happens to the rest of the light?

(c) What is the transmission at critical coupling for detuning $\Delta = \kappa/2$?

Problem 2 If a nanoparticle (10–100 nm, e.g. a small colloid or a virus particle) attaches to the toroid surface, it can disturb the electric field distribution in the toroid and scatter some of the light. The particle can be detected by looking at the frequency shift of a resonance when the particle binds to the toroid. However, also temperature fluctuations can cause the resonance to shift, so this method suffers from thermal noise and other environmental disturbances.

Another way to detect the particle is to look at *mode splitting*: because the particle scatters the light, it can scatter light from the clockwise-propagating WGM into the counterclockwise-propagating WGM. To account for a scattering with strength γ, we can add another term in the equations of motion for the intra-cavity amplitudes:

$$\frac{\mathrm{d}}{\mathrm{d}t}a_\downarrow(t) = (\mathrm{i}\Delta - \kappa/2)a_\downarrow(t) + \mathrm{i}\frac{\gamma}{2}a_\uparrow(t) + \sqrt{\kappa_{\mathrm{ex}}}s_{\mathrm{in}}(t) \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a_\uparrow(t) = (\mathrm{i}\Delta - \kappa/2)a_\uparrow(t) + \mathrm{i}\frac{\gamma}{2}a_\downarrow(t). \tag{2}$$

Due to the coupling between these equations $a_\downarrow$ and $a_\uparrow$ are no longer normal modes of the system.

(a) Show that, with a suitable change of basis, the above system of coupled equations of motion can be rewritten as a set of *uncoupled* equations for the new normal modes a_+ and a_-, which will be a linear combination of $a_\downarrow$ and $a_\uparrow$. What are these *normalised* modes in terms of $a_\downarrow$ and $a_\uparrow$?

(b) Let $\Delta_\pm = \omega - \omega_\pm$ be the detunings from the resonance frequencies of the new modes $a_\pm$. What are the resonance frequencies $\omega_\pm$ in terms of ω_c and the intensity damping rates $\kappa_\pm$ of the normal modes $a_\pm$?

Problem 3 In the regime where the splitting is far larger than the decay rate $\gamma \gg \kappa$, the resonances of the individual modes can be approximated as seperate Lorentzians. Considering the resonance ω_-, the expression of transmission with respect to the detuning Δ_- is

$$T = 1 - \frac{3}{4}\frac{(\kappa/2)^2}{(\kappa/2)^2 + \Delta_-^2}.$$

In the limit of small volumes, the splitting γ (and thus $\omega_\pm$) is directly proportional to the volume V of the particle. If a laser is tuned at a well-known frequency on the side of the resonance ω_-, its detuning Δ_- will also depend on the volume V.

(a) Assuming an input laser power P_{in} and a detector integration time τ, derive the expression of the derivative of the transmitted photons with respect to the particle volume as a function of Δ_-

$$\frac{\partial N_{\mathrm{out}}}{\partial V}(\Delta_-).$$

This shows that by monitoring the transmittance of a continuous-wave laser through the fiber in time, small changes to the nanoparticle diameter can be detected, for example those caused by evaporation of material from the nanoparticle.

Assume input power $P_{\mathrm{in}} = 1\,\mu\mathrm{W}$, integration time $\tau = 1\,\mu\mathrm{s}$, and input laser wavelength in vacuum $\lambda = 670\,\mathrm{nm}$. A spherical sulfur nanoparticle of radius $r = 15\,\mathrm{nm}$ adheres to the surface of the toroid ring, causing a splitting of $\gamma/2\pi = 1\,\mathrm{GHz}$.

(b) If the laser is detuned at $\Delta_- = \kappa/2$ and the output noise is dominated by its shot noise $\sqrt{N_{\mathrm{out}}}$, find the minimum change in volume that can be resolved with this apparatus if the quality factor is $Q = 1 \times 10^8$.

(c) Knowing that the atomic weight of sulfur is $m_{\mathrm{S}} = 32\,\mathrm{u}$ and its density is $\rho_{\mathrm{S}} = 2000\,\mathrm{kg\,m^3}$, what is the minimal number of atoms we can observe evaporate from the nanoparticle?